NPS55-90-03

NAVAL POSTGRADUATE SCHOOL

Monterey, California

ITEM IDENTITIES AND THEIR RELATED OBSERVABLES

DONALD P. GAVER
PATRICIA A. JACOBS

November 1989

Approved for public release; distribution is unlimited.

Prepared for:
Office of Naval Research
Arlington, VA 22217

NAVAL POSTGRADUATE SCHOOL,
MONTEREY, CALIFORNIA

Rear Admiral R. W. West, Jr.
Superintendent

Harrison Shull
Provost

This report was prepared for Office of Naval Research and funded by the Naval Postgraduate School.

This report was prepared by:

ASSIFIED

RITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

EPORT SECURITY CLASSIFICATION: NCLASSIFIED	1b RESTRICTIVE MARKINGS
ECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution is unlimited.
ECLASSIFICATION/DOWNGRADING SCHEDULE	
RFORMING ORGANIZATION REPORT NUMBER(S): PS55-90-03	5 MONITORING ORGANIZATION REPORT NUMBER(S)
AME OF PERFORMING ORGANIZATION: aval Postgraduate School; 6b OFFICE SYMBOL (If applicable): 55	7a NAME OF MONITORING ORGANIZATION
DDRESS (City, State, and ZIP Code): Ionterey, CA 93943-5000	7b ADDRESS (City, State, and ZIP Code)
AME OF FUNDING/SPONSORING RGANIZATION: Office of Naval Research; 8b OFFICE SYMBOL (If applicable): 430	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: O&MN, Direct Funding
DDRESS (City, State, and ZIP Code): rlington, VA 22217	10 SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO; PROJECT NO; TASK NO; WORK UNIT ACCESSION NO

ITLE (Include Security Classification)
TEM IDENTITIES AND THEIR RELATED OBSERVABLES

ERSONAL AUTHOR(S)
Gaver, Donald P., Jacobs, Patricia A.

TYPE OF REPORT	13b. TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
echnical	FROM ____ TO ____	1989, November	28

UPPLEMENTARY NOTATION

COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Bayesian paradigm, autoregressive process, classification

BSTRACT (Continue on reverse if necessary and identify by block number)

Each of J items has a characteristic Signature which varies in time. At time 0, the value of a Signature and the identity of the corresponding item are known. No further values of Signatures are observed until a later time t. At time t, a Signature from an unknown item is observed. The problem is to estimate the identity of the item whose Signature is observed at time t. The estimation procedure studied is to estimate the identity of the unknown item to be that one which maximizes the posterior probability of producing the observed Signature.

DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
NAME OF RESPONSIBLE INDIVIDUAL: Donald P. Gaver	22b TELEPHONE (Include Area Code): (408) 646-2605; 22c OFFICE SYMBOL: 55Gv

FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

Printing Office 1986—606-243

ITEM IDENTITIES AND THEIR RELATED OBSERVABLES

D. P. Gaver
P. A. Jacobs

Abstract

Each of J items has a characteristic Signature which varies in time. At time 0, the value of a Signature and the identity of the corresponding item are known. No further values of Signatures are observed until a later time t>0. At time t, a Signature from an unknown item is observed. The problem is to estimate the identity of the item whose Signature is observed at time t. The estimation procedure studied is to estimate the identity of the unknown item to be that one which maximizes the posterior probability of producing the observed Signature.

Key Words: Classification; Bayesian Paradigm; autoregressive process

ITEM IDENTITIES AND THEIR RELATED OBSERVABLES

D. P. Gaver
P. A. Jacobs

1. The Problem

We are concerned with a diagnostic problem that may occur in many applied areas, with medicine and mechanical system reliability furnishing handy examples.

Suppose there are J Items (e.g., diseases or faults), each of which has a characteristic Signature which varies in time; the Signature of Item i is

$$Y_i(t) = \theta_i + X_i(t) \qquad \begin{array}{l} i = 1,2,\ldots,J \\ t = 0,1,2,\ldots \end{array} \tag{1.1}$$

For the moment $\{X_i(t)\}$ is an unspecified univariate stochastic process, but one that loosely, has the property of staying near θ_i in finite time, and for convenience has some stationary or steady-state behavior. One could think of $Y_i(t)$ as a physical index characteristic of a particular disease or disability, e.g., blood pressure, heart-beat pattern, mechanical vibration spectrum, etc.

Suppose that $Y_i(t)$ is only evident or observable occasionally at times unrelated to the magnitude of $Y_i(t)$, but driven by other forces (this is a bit implausible in some situations because a high blood pressure or temperature may actually induce medical observation; the formulation can be changed to reflect this later). Suppose that the *Signature and the identity of an item* are both observed at time t=0. Suppose that later on, at time t, only the Signature of an item is observed. The *first question* is: what is the probability that, given the Signature value observed, its originating item is any particular one of a list of candidates, e.g., i=13 out of a particular list j=1, 2, ... 13, ... J = 39?

2. Question: Who is Being Observed?

Here we become quite specific, assuming for illustration that the Signature of Item j is AR(1):

$$Y_j(t) = \theta_j + X_j(t), \tag{2.1}$$

where

$$X_j(t) = \rho_j X_j(t-1) + \varepsilon_j(t), \quad t = 0, 1, 2, \ldots, 17, \ldots \tag{2.2}$$

with $\{\varepsilon_j(t)\} \sim$ IID N $(0,\sigma_j^2)$. For the moment assume that the basic constants θ_j, $j = 1, 2, \ldots, J$ are known, as are the correlations $|\rho_j| \leq 1$ and the variances σ_j^2.

If $|\rho_j| < 1$ then, given $X_j(0)$, $X_j(t)$ is Normal/Gauss

$$X_j(t) \mid X_j(0) \sim N\left(X_j(0)\rho_j^t, \sigma_j^2(t) \equiv \sigma_j^2 \left[\frac{1-\rho_j^{2t}}{1-\rho_j^2} \right] \right) \tag{2.3}$$

or, in terms of the actual observable $Y_j(t)$,

$$Y_j(t) \mid Y_j(0) \sim N((Y_j(0)-\theta_j)\rho_j^t + \theta_j, \sigma_j^2(t)) \tag{2.4}$$

Apparently if t runs on indefinitely, its distribution approaches

$$N\left(\theta_j, \sigma_j^2(\infty)\right), \text{ with } \sigma_j^2(\infty) = \sigma_j^2 / \left(1-\rho_j^2\right) \tag{2.5}$$

Scenario: Suppose there are potentially J items around, but admit the possibility that each can vanish, so that there is some probability, which may be less than one, that the item will actually be "around" at time t. Let the probability that one, the jth, is present at t be $p_j(t)$. Next, let $s_j(t)$ be the probability that the jth be observable, e.g., is emitting sounds, given that it is present; assume independence between items. All of this is preliminary, and can (and should) be modified as required. Now consider that event

A) Item i has been observed at t=0, and its Signature is observed/noted to be $Y_i(0) = y(0)$.

If it has been a long time since an observation was made then it is plausible to take the probability of the above event as

$$P\{Y_i(0)\,\varepsilon(dy(0)),\, I(0) = i\} = p_i(0)\, s_i(0)\varphi\left(\frac{y(0)-\theta_i}{\sigma_i(\infty)}\right)\frac{1}{\sigma_i(\infty)}\,dy(0) \tag{2.6}$$

φ being the N(0,1) density; that is, we observe Item i after his Signature is in statistical equilibrium, and measure time from that (such an) instant.

Time t elapses and then for any j (either the same as i, or different)

B) Item j is observed at time t>0, with Signature value $Y_j(t) = y(t)$.

Since the AR(1) process is Markov, we can write down the joint probability

$$P\{Y_i(0)\,\varepsilon(dy(0)),\, I(0) = i,\, Y_j(t)\,\varepsilon(dy(t)),\, I(t) = j\} =$$

$$p_i(0)\, s_i(0)\, \varphi\left(\frac{y(0)-\theta_i}{\sigma_i(\infty)}\right)\cdot\frac{dy(0)}{\sigma_i(\infty)}\cdot p_j(t)\, s_j\,(t)\, \varphi\left(\frac{y(t)-m_j(t)}{\sigma_j(t)}\right)\frac{dy(t)}{\sigma_j(t)} \tag{2.7}$$

where we can think of $p_j(t)$ as the *conditional* probability that j is actually eligible for observation, and $s_j(t)$ as the conditional probability of being observable. Note that it is natural for $j \neq i$ that

$$m_j(t) = \theta_j \tag{2.8,a}$$

$$\sigma_j(t) = \sigma_j(\infty) = \sigma_j^2 \,/\left(1-\rho_j^2\right) \tag{2.8,b}$$

for at t it is still a "long time" since $j(\neq i)$ was last observed; whereas if j=i, the same item that was observed and identified at t=0, then

$$m_i(t) = \theta_i + (y(0)-\theta_i)\,\rho_i^t \tag{2.9,a}$$

and

$$\sigma_i^2(t) = \sigma_i^2\left(\frac{1-\left(\rho_i^2\right)^t}{1-\rho_i^2}\right). \tag{2.9,b}$$

It follows that

C) the conditional probability of observing a particular item and observing it with Signature value y(t) *given* the initial condition $C_i(0) = \{Y_i(0), I(0)\}$ is just

$$P\left\{Y_j(t)\ \varepsilon(dy(t)),\ I(t) = j \mid C_i(0) \equiv (y(0), I(0))\right\} =$$

$$p_j(t)\ s_j(t)\ \varphi\left(\frac{y(t) - m_j(t)}{\sigma_j(t)}\right)\frac{dy(t)}{\sigma_j(t)} \tag{2.10}$$

where the appropriate $m_j(t)$ and $\sigma_j(t)$ come from (2.8) or (2.9). In words, (2.10) provides the probability of *seeing* or positively identifying a particular item, j, and simultaneously *measuring* its current Signature value, given such a complete observation at t=0.

Next

$$\text{D)}\ P\left\{Y(t)\ \varepsilon(dy(t)) \mid C_i(0)\right\} = \sum_{j=1}^{J} P\left\{Y_j(t)\varepsilon(dy(t)),\ I(t) = j \mid C_i(0)\right\}$$

$$= \sum_{j=1}^{J} p_j(t)s_j(t)\ \varphi\left(\frac{y(t) - m_j(t)}{\sigma_j(t)}\right)\frac{dy(t)}{\sigma_j(t)}. \tag{2.11}$$

This is the probability that one *measures* a Signature at t, but cannot make a complete identification.

From (2.10) and (2.11) we can calculate the probability of the identity of the item whose signature is measured at time t, given the last complete observation:

$$\text{E)}\ P\{I(t) = j \mid Y(t) = y(t), C_i(0)\} = \frac{p_j(t)\ s_j(t)\ \varphi\left(\frac{y(t)-m_j(t)}{\sigma_j(t)}\right)\cdot\frac{1}{\sigma_j(t)}}{\sum_{k=1}^{J} p_k(t)\ s_k(t)\ \varphi\left(\frac{y(t) - m_k(t)}{\sigma_k(t)}\right)\cdot\frac{1}{\sigma_k(t)}} \tag{2.12}$$

If one looks hard at (2.12) it seems clear that when $\rho_i>0$ and y(t) is near y(0), *and* j=i, the above probability becomes closer and closer to unity as t decreases, while if $j \neq i$ that probability (that Item $j \neq i$) correspondingly decreases.

3. The Probability of Making the Wrong Decision

In this section we study properties of making a decision based on choosing the item which maximizes the posterior probability of a particular item being the one whose Signature is observed at time t given the last complete observation at time 0. For simplicity we will assume that there are two items whose parameters are θ_1 and θ_2.

3.1 The parameters θ_1 and θ_2 are known.

Suppose the parameters θ_1 and θ_2 are known and $|\rho_i|<1$. The conditional distribution of Y(t) given Y(0) = y(0), I(0) = 1, I(t) = 1 is normal with mean

$$m_1(t) = \theta_1 + (y(0)-\theta_1)\rho_1^t \tag{3.1}$$

and standard deviation

$$\sigma_1(t) = \sigma_1(\infty)\sqrt{1-\rho_1^{2t}} \tag{3.2}$$

Further,

$$\begin{aligned} \frac{Y(t)-m_2(t)}{\sigma_2(t)} &= \frac{Y(t)-m_1(t) + (m_1(t)-m_2(t))}{\sigma_2(t)} \qquad (3.3) \\ &= \frac{\sigma_1(t)}{\sigma_2(t)}\,\frac{Y(t)-m_1(t)}{\sigma_1(t)} + \frac{m_1(t)-m_2(t)}{\sigma_2(t)} \end{aligned}$$

where $m_2(t)$ and $m_1(t)$ are given by (2.8).

Assume that item j will be identified with the Signature at time t if it maximizes (2.12); then,

$$P\left\{\text{wrong item is identified} \mid I(0) = 1, I(t) = 1, Y(0) = y(0)\right\} \tag{3.4}$$

$$= P\left\{ \begin{array}{l} p_2(t)s_2(t)\,\dfrac{1}{\sqrt{2\pi}\,\sigma_2(t)}\,\exp\left\{-\dfrac{1}{2}\,\dfrac{\left(Y(t)-m_2(t)\right)^2}{\sigma_2(t)^2}\right\} \\ > p_1(t)s_1(t)\,\dfrac{1}{\sqrt{2\pi}\,\sigma_1(t)}\,\exp\left\{-\dfrac{1}{2}\,\dfrac{\left(Y(t)-m_1(t)\right)^2}{\sigma_1(t)^2}\right\} \end{array} \;\middle|\; I(0)=1, I(t)=1, Y(0)=y(0) \right\}$$

$$= P\left\{\frac{p_2(t)s_2(t)}{p_1(t)s_1(t)}\,\frac{\sigma_1(t)}{\sigma_2(t)} > \exp\left\{-\frac{1}{2}Z^2 + \frac{1}{2}\left(c(t)Z + d(t)\right)^2\right\}\right\} \tag{3.5}$$

where Z has a standard normal distribution

$$c(t) = \frac{\sigma_1(t)}{\sigma_2(t)} = \sqrt{1-\rho_1^{2t}}\;\frac{\sigma_1(\infty)}{\sigma_2(\infty)}$$

and

$$d(t) = \frac{m_1(t) - m_2(t)}{\sigma_2(t)} = \frac{(y(0)-\theta_1)\rho_1^t + (\theta_1 - \theta_2)}{\sigma_2(\infty)}$$

$$P\left\{\text{wrong decision} \mid I(0) = 1, I(t)=1, Y(0) = y(0)\right\}$$

$$= P\left\{\frac{p_2(t)s_2(t)}{p_1(t)s_1(t)}\,\frac{\sqrt{1-\rho_1^{2t}}\;\sigma_1(\infty)}{\sigma_2(\infty)} > \exp\left\{-\frac{1}{2}\left[(1-c(t)^2)Z^2 - 2c(t)d(t)Z - d(t)^2\right]\right\}\right\}$$

$$= P\left\{-2\ln\left[\frac{p_2(t)s_2(t)}{p_1(t)s_1(t)}\,\frac{\sqrt{1-\rho_1^{2t}}\;\sigma_1(\infty)}{\sigma_2(\infty)}\right] < (1-c(t)^2)Z^2 - 2c(t)d(t)Z - d(t)^2\right\}$$

$$= P\left\{0 < (1-c(t)^2)Z^2 - 2c(t)d(t)Z - d(t)^2 + 2\ln\left[\frac{p_2(t)s_2(t)}{p_1(t)s_1(t)}\sqrt{1-\rho_1^{2t}}\;\frac{\sigma_1(\infty)}{\sigma_2(\infty)}\right]\right\} \tag{3.6}$$

Note that as $t \to 0$ $\ln\left[1-\rho_1^{2t}\right] \to -\infty$ and thus

$$P\left\{\text{wrong decision} \mid I(0) = 1, I(t) = 1, Y(0) = y\right\} \to 0.$$

Example: Suppose $\sigma_i = \sigma$ $i = 1,2$, $|\rho_i| = |\rho| < 1$, $i=1,2$ and $p_1(t)s_1(t) = p_2(t)s_2(t)$; then

$$P\left\{\text{wrong decision} \mid I(0) = 1, I(t) = 1, Y(0) = y(0)\right\}$$

$$= P\left\{0 < (1-c(t)^2)Z^2 - 2c(t)d(t)Z - d(t)^2 + 2\ln c(t)\right\} \tag{3.7}$$

with

$$c(t) = \sqrt{1-\rho^{2t}} \tag{3.8,a}$$

and

$$d(t) = \frac{(y(0) - \theta_1)\rho^t + (\theta_1-\theta_2)}{\sigma/\sqrt{1-\rho^2}}. \tag{3.8,b}$$

As $t\to\infty$, $d(t)\to\dfrac{(\theta_1-\theta_2)}{\sigma/\sqrt{1-\rho^2}}$, $c(t)\to1$ and

$$P\{\text{wrong decision} \mid I(0) = 1, I(t) = 1, Y(0) = y(0)\}$$

$$\to \left[1-\Phi\left(\frac{|d(\infty)|}{2}\right)\right] \tag{3.9}$$

where Φ is the standard normal distribution function. In this case, the conditional probability of making a wrong decision decreases as $|\theta_1-\theta_2|$ increases and the probability increases as σ increases.

For arbitrary $t > 0$ such that $c\neq1$,

$$P\{\text{wrong decision} \mid I(0) = 1, I(t) = 1, Y(0) = y(0)\}$$

$$= P\left\{\left(Z-\frac{c(t)\, d(t)}{1-c(t)^2}\right)^2 > \frac{d(t)^2}{(1-c(t)^2)^2} - \frac{2 \ln c(t)}{1-c(t)^2}\right\}$$

$$= P\left\{\left|Z-\frac{c(t)\, d(t)}{1-c(t)^2}\right| > \sqrt{\frac{d(t)^2}{(1-c(t)^2)^2} - \frac{2 \ln c(t)}{1-c(t)^2}}\right\}$$

$$= P\left\{\left|Z-\frac{c(t)\, d(t)}{1-c(t)^2}\right| > \sqrt{\frac{d(t)^2}{\rho^{4t}} - \frac{\ln(1-\rho^{2t})}{\rho^{2t}}}\right\}$$

$$= 1 - \Phi\left(\sqrt{1-\rho^{2t}}\ \rho^{-2t}d(t) + k(t)\right) + \Phi\left(\sqrt{1-\rho^{2t}}\ \rho^{-2t}d(t) - k(t)\right) \tag{3.10a}$$

where

$$k(t) = \left[d(t)^2\, \rho^{-4t} - \rho^{-2t} \ln\left(1-\rho^{2t}\right)\right]^{\frac{1}{2}}.$$

Note that as $t\to0$, $c(t)\to0$ and the conditional probability of an incorrect decision tends to zero.

A numerically more stable form of (3.10a) is the following.

$$P\{\text{wrong decision} \mid I(0)=1, I(t)=1, Y(0)=y(0)\} = 1-\Phi(A_1)+\Phi(A_2) \qquad (3.10b)$$

where

$$A_1 = [-d(t)^2 + 2\ln c(t)] \left[c(t)d(t) - \sqrt{d(t)^2 - (1-c(t)^2)2\ln c(t)}\right]^{-1}$$

$$A_2 = [-d(t)^2 + 2\ln c(t)] \left[c(t)d(t) + \sqrt{d(t)^2 - (1-c(t)^2)2\ln c(t)}\right]^{-1}$$

The unconditional probability of a wrong classification can be found by taking the expected value of (3.10b) with respect to the normal random variable Y(0) with mean θ_1 and standard deviation $\sigma(\infty)$. The evaluation of the integrals is performed numerically using Gauss-Hermite quadration with 59 points; cf. A. H. Stroud et al [1966] and Naylor and Smith [1982]. Figures 1 and 2 show the unconditional probability of wrong classification at time 1, 2, ..., 100 for parameter values $\theta_1 = 1$, $\theta_2 = 2$, $\sigma = 1$ and values of $\rho = 0, 0.5, 0.8, 0.90, 0.95$. (resp. curves A, B, C, D, E).

Figure 1 shows the probability of wrong classification. As noted above, the probability of wrong classification is small for small values of t and then increases to a limiting value. For very small times having a larger ρ results in smaller probabilities of error. It is seen from (3.9) that the limiting value is a function of $\sigma(\infty) = \sigma/\sqrt{1-\rho^2}$. Hence the larger ρ is the larger the limiting probability of wrong classification. Thus, while for very small times, having a large positive ρ decreases the probability of wrong classification, at larger time it increases the probability of wrong classification.

In Figure 2, each probability of wrong classification is divided by its asymptotic value. These normalized probabilities are between 0 and 1. The fraction (relative to the limiting probability) of the probability of wrong classification for each time t is monotone in ρ. For a fixed time t, for a high ρ, the fraction of the probability of wrong classification is less than or equal to that for a lower ρ. The higher ρ is, the longer it takes for the probability of error to reach its asymptotic value.

3.2. The parameters θ_1 and θ_2 have noninformative priors.

In this subsection we will assume that θ_1 and θ_2 have noninformative prior distributions. Thus, assuming $|\rho_i|<1$,

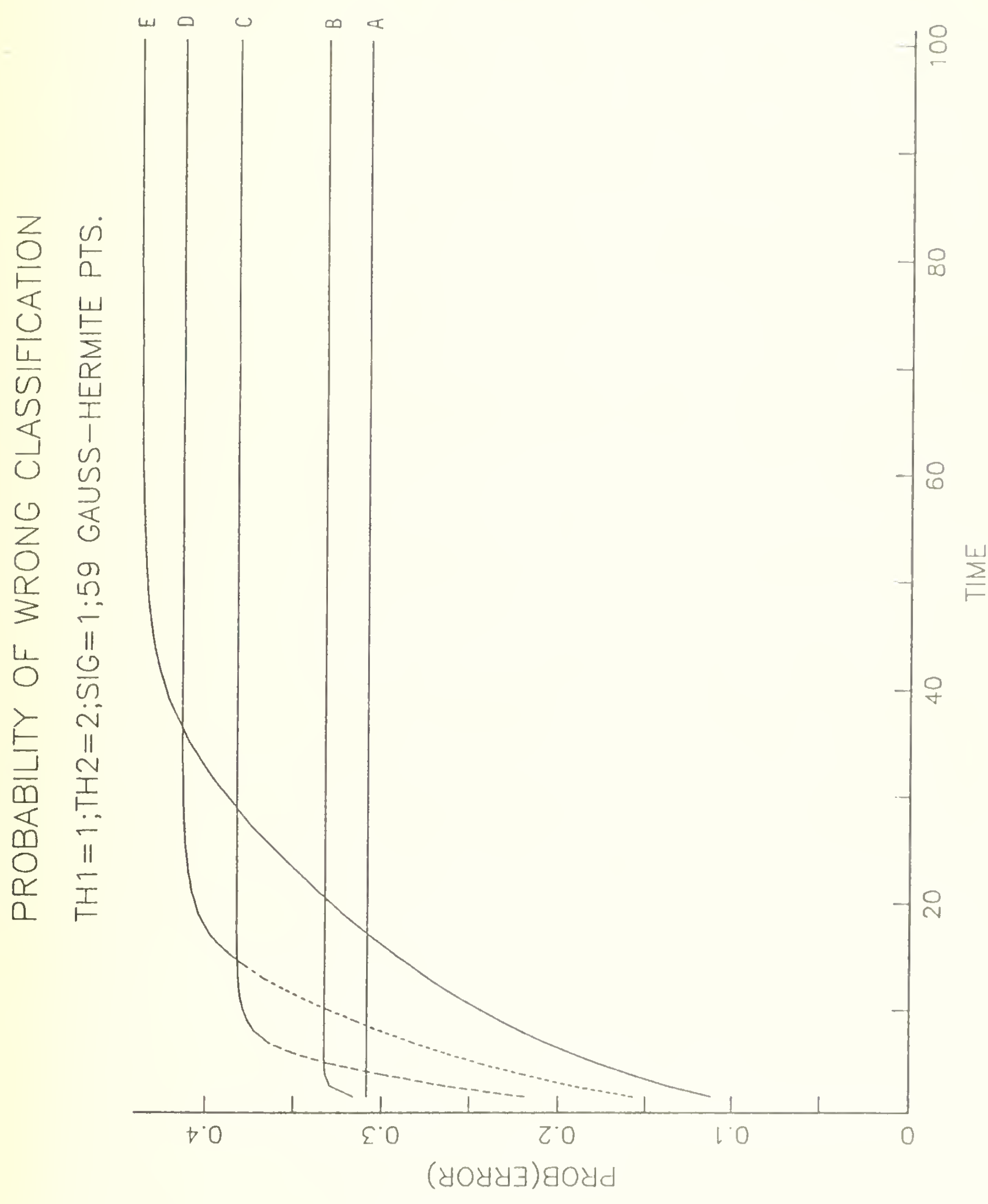
PROBABILITY OF WRONG CLASSIFICATION
TH1=1;TH2=2;SIG=1;59 GAUSS-HERMITE PTS.
PROB(ERROR)
0
0.1
0.2
0.3
0.4
TIME
20
40
60
80
100
A
B
C
D
E

Figure 1

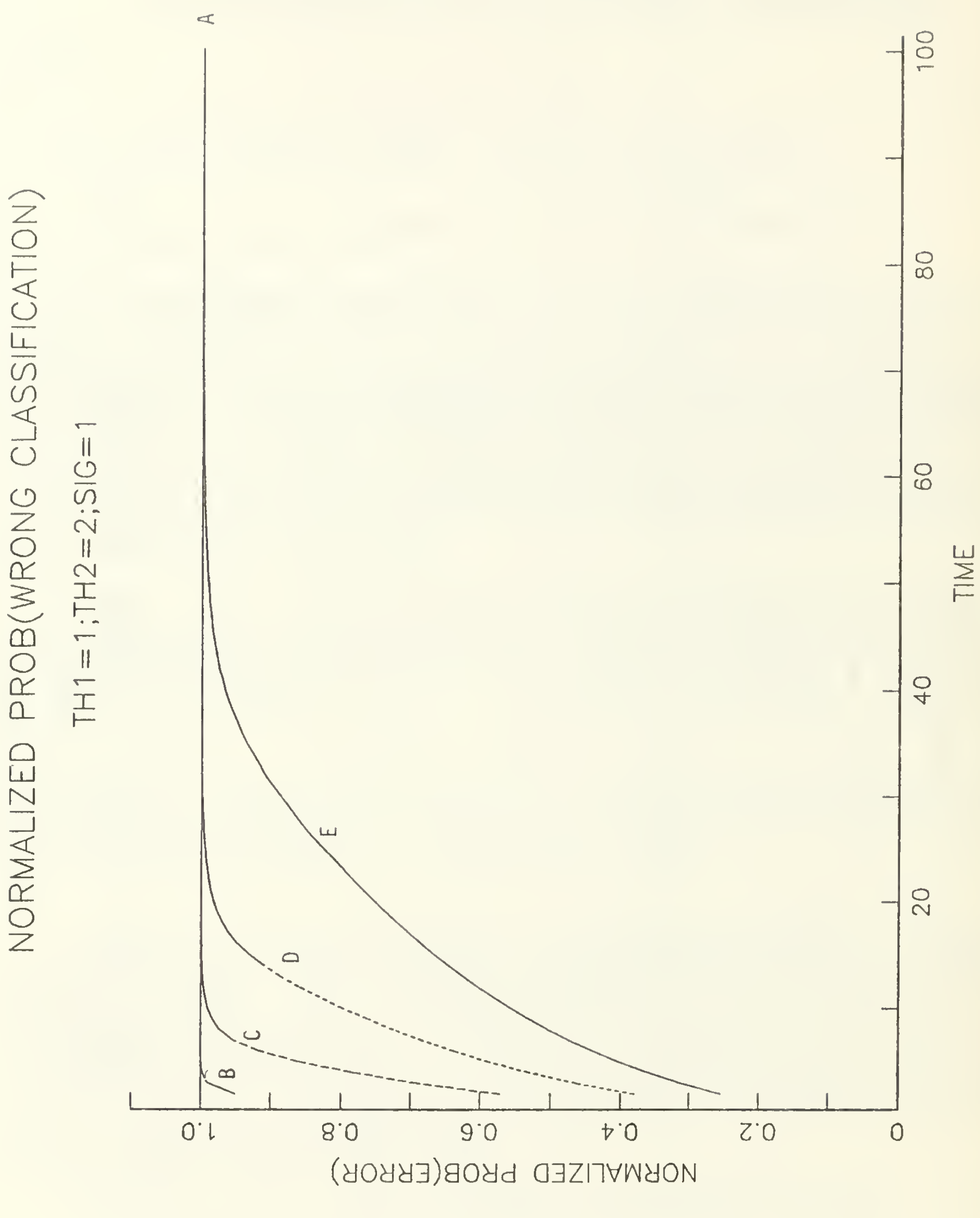
NORMALIZED PROB(WRONG CLASSIFICATION)
TH1=1;TH2=2;SIG=1
NORMALIZED PROB(ERROR)
0
0.2
0.4
0.6
0.8
1.0
TIME
20
40
60
80
100
A
B
C
D
E

Figure 2

$$P\left\{Y(0)\,\varepsilon dy(0),\,\theta\varepsilon d\theta \mid I(0)=1\right\}$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma_1(\infty)} \exp\left\{-\frac{1}{2}\left(y(0)-\theta\right)^2/\sigma_1(\infty)^2\right\} dy(0)d\theta \,. \tag{3.11}$$

Hence, formally for a non-informative improper prior,

$$P\left\{Y(0)\varepsilon dy(0) \mid I(0)=1\right\} = 1\cdot dy(0) \tag{3.12}$$

and thus

$$P\left\{\theta\varepsilon d\theta \mid I(0)=1,\, Y(0)=y(0)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1(\infty)} \exp\left\{-\frac{1}{2}\left(\theta - y(0)\right)^2/\sigma_1(\infty)^2\right\} d\theta \,. \tag{3.13}$$

Further,

$$P\left\{Y(t)\varepsilon dy(t),\,\theta\varepsilon d\theta \mid I(0)=1,\, I(t)=1,\;\; Y(0)=y(0)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1(t)} \exp\left\{-\frac{1}{2}\left(y(t)-\theta-\rho_1^t\left(y(0)-\theta\right)\right)^2/\sigma_1(t)^2\right\}$$

$$\times \frac{1}{\sqrt{2\pi}\,\sigma_1(\infty)} \exp\left\{-\frac{1}{2}\left(\theta - y(0)\right)^2/\sigma_1(\infty)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1(t)\sqrt{2\pi}\,\sigma_1(\infty)} \exp\left\{-\frac{1}{2}\left(\theta-\mu_1(t)\right)^2/v_1(t)^2 - \frac{1}{2}R\right\} \tag{3.14}$$

where the usual calculations yield

$$v_1(t)^2 = \left[\frac{\left(1-\rho_1^t\right)^2}{\sigma_1(t)^2} + \frac{1}{\sigma_1(\infty)^2}\right]^{-1} = \left[\frac{2\left(1-\rho_1^t\right)}{\left(1-\rho_1^{2t}\right)\sigma_1(\infty)^2}\right]^{-1} \tag{3.15}$$

$$\frac{\mu_1(t)}{v_1(t)^2} = \left(y(t) - \rho_1^t\, y(0)\right)\frac{\left(1-\rho_1^t\right)}{\sigma_1(t)^2} + y(0)/\sigma_1(\infty)^2 \tag{3.16}$$

$$R = \frac{v_1(t)^2}{\sigma_1(\infty)^2 \sigma_1(t)^2}\left[y(t) - y(0)\right]^2 = \left[y(t)-y(0)\right]^2/2\left[1-\rho_1^t\right]\sigma_1(\infty)^2 \tag{3.17}$$

$$\sigma_1(\infty)^2 = \sigma_1^2 / \left(1-\rho_1^2\right) \tag{3.18}$$

$$\sigma_1(t)^2 = \left(1-\rho_1^{2t}\right)\sigma_1(\infty)^2 \tag{3.19}$$

Hence,

$$P\left\{Y(t)\varepsilon dy(t) \mid I(0) = 1, I(t) = 1, Y(0) = y(0)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2\left[1-\rho_1^t\right]\sigma_1(\infty)^2}} \exp\left\{-\frac{1}{2}(y(t) - y(0))^2 / 2\left[1-\rho_1^t\right]\sigma_1(\infty)^2\right\} \tag{3.20,a}$$

Similarly

$$P\left\{Y(t)\varepsilon dy(t), \theta\varepsilon d\theta \mid I(t)=2, I(0)=1, Y(0)=y(0)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_2(\infty)} \exp\left\{-\frac{1}{2}(y(t)-\theta)^2/\sigma_2(\infty)^2\right\} dy(t)d\theta$$

and

$$P\left\{Y(t)\varepsilon dy(t) \mid I(t)=2, I(0)=1, Y(0)=y(0)\right\} = 1\cdot dy(t) \tag{3.20,b}$$

where $\sigma_2(\infty) = \sigma_2 / \left(1-\rho_2^2\right)$.

Suppose item j is estimated to be the one observed at time t if it maximizes

$$P\left\{I(t)=j \,\middle|\, I(0)=1, Y(0) = y(0), Y(t)=y(t)\right\}.$$

It follows that

$$P\left\{\text{Item 2 is chosen} \mid I(0)=1, Y(0)=y(0), I(t)=1\right\}$$

$$= P\left\{p_2(t)s_2(t) > p_1(t)s_1(t)\ \frac{1}{\sqrt{2\pi}\ \alpha_1(t)}\ \exp\left\{-\frac{1}{2}(Y(t)-y(0))^2/\alpha_1(t)^2\right\} \,\Big|\, I(0)=1, I(t)=1, Y(0)=y(0)\right\}$$

$$= P\left\{\sqrt{2\pi}\ \sigma_1(\infty)\ \frac{p_2(t)s_2(t)}{p_1(t)s_1(t)}\ \sqrt{2\left(1-\rho_1^t\right)} > \exp\left\{-\frac{1}{2}Z^2\right\}\right\}$$

$$= P\left\{-2\ \ln\left[\sqrt{2\pi}\ \frac{\sigma_1}{\sqrt{1-\rho_1^2}}\ \frac{p_2(t)s_2(t)}{p_1(t)s_1(t)}\ \sqrt{2(1-\rho_1^t)}\right] < Z^2\right\} \tag{3.21}$$

where Z is a standard normal random variable and $\alpha_1(t) = \left(1-\rho_1^t\right)\sigma_1(\infty)$.

Note that as $t \to 0$, the probability of choosing the wrong object tends to 0. If

$$k(t) \equiv \sqrt{2\pi}\ \frac{\sigma_1}{\sqrt{1-\rho_1^2}}\ \frac{p_2(t)s_2(t)}{p_1(t)s_1(t)}\ \sqrt{2\left(1-\rho_1^t\right)} > 1 \tag{3.22}$$

then the conditional probability of making the wrong decision is 1.

If $k(t) < 1$ then

$P\{\text{Item 2 is chosen} \mid I(0)=1, Y(0)=y(0), I(t)=1\}$

$$= 2\left[1-\Phi(-2\ \ln(k(t)))\right] \tag{3.23}$$

where Φ is the standard normal cdf. Note that (3.23) is independent of y(0). As σ_1 increases, the conditional probability of making the wrong decision increases. Similarly as t increases the probability of making the wrong decision increases. As $p_1(t)s_1(t)$ decreases, the probability of making the wrong decision also increases.

In the case in which $p_1(t)\, s_1(t) = p_2(t)\, s_2(t)$, $\rho_1 = \rho_2 = \rho > 0$, and $\sigma_1 = \sigma_2 = \sigma$

$$k(t) = 2\sqrt{\pi} \cdot \left[\frac{1-\rho^t}{1-\rho^2}\right]^{\frac{1}{2}}.$$

For $t = 1$, $k(t)$ and hence the probability of error decreases as ρ increases. For $t = 2$, $k(t) = \sqrt{2\pi}\,\sigma$ for all values of ρ and so the probability of choosing the wrong item increases as a function of t. The limiting value of $k(t)$ as $t \to \infty$ once again depends on $\sigma(\infty) = \sigma/\sqrt{1-\rho^2}$. Hence as ρ increases the limiting probability as $t \to \infty$ of choosing the wrong item increases. Figure 3 displays the probability of choosing the wrong item for $\sigma = 0.1$ and $\rho = 0, 0.5, 0.8, 0.9, 0.95$ (resp. curves A, B, C, D, E). Note that for $t>2$, it is $\rho = 0$ that has the smallest probability of incorrect identification. These results depend on the specific prior (3.20,b). In the next subsection we will consider Gaussian priors.

3.3 The parameters θ_1 and θ_2 have Gaussian priors

In this section we will assume θ_i has a normal prior with mean μ_i and variance τ_i^2, $i = 1,2$. Thus, assuming $|\rho_i| < 1$,

$$P\left\{Y(0)\varepsilon dy(0),\ \theta\varepsilon d\theta \mid I(0)=1\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1(\infty)} \frac{1}{\sqrt{2\pi}\,\tau_1} \exp\left\{-\frac{1}{2}\frac{(y(0)-\theta)^2}{\sigma_1(\infty)^2} - \frac{1}{2}\frac{(\theta-\mu_1)^2}{\tau_1^2}\right\} dy(0)d\theta \tag{3.24}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1(\infty)} \frac{1}{\sqrt{2\pi}\,\tau_1} \exp\left\{-\frac{1}{2}\frac{(\theta-m_1(0))^2}{v_1(0)^2} - \frac{1}{2} R_1(0)\right\} dy(0)d\theta \tag{3.25}$$

where straightforward calculations yield

$$v_1(0)^{-2} = \frac{1}{\tau_1^2} + \frac{1}{\sigma_1(\infty)^2} = \frac{\sigma_1(\infty)^2 + \tau_1^2}{\tau_1^2\,\sigma_1(\infty)^2} \tag{3.26}$$

$$m_1(0) = \left[\left(y(0)/\sigma_1(\infty)^2\right) + \left(\mu_1/\tau_1^2\right)\right] v_1(0)^2$$

$$= \frac{\tau_1^2\, y(0) + \sigma_1(\infty)^2 \mu_1}{\sigma_1(\infty)^2 + \tau_1^2} \tag{3.27}$$

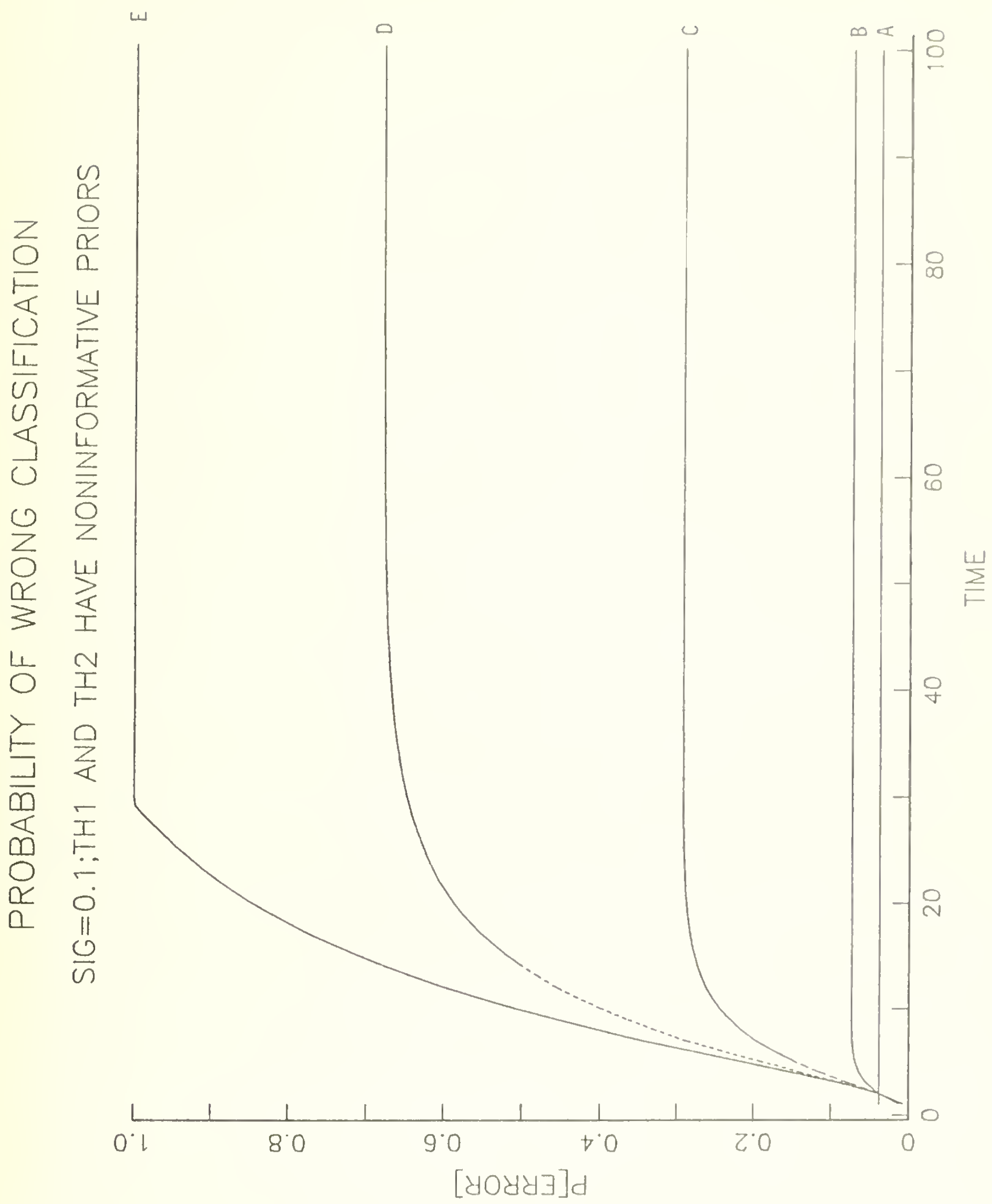
PROBABILITY OF WRONG CLASSIFICATION
SIG=0.1;TH1 AND TH2 HAVE NONINFORMATIVE PRIORS
P[ERROR]
0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100
TIME
A
B
C
D
E

Figure 3

$$R_1(0) = \frac{1}{\sigma_1(\infty)^2 + \tau_1^2}\,(y(0) - \mu_1)^2. \tag{3.28}$$

with

$$\sigma_1(\infty)^2 = \sigma_1^2 / \left(1-\rho_1^2\right)$$

Therefore,

$$P\left\{\theta\varepsilon d\theta \mid I(0)=1,\, Y(0)=y(0)\right\} = \frac{1}{\sqrt{2\pi}\, v_1(0)} \exp\left\{-\frac{1}{2}\frac{\left(\theta - m_1(0)\right)^2}{v_1(0)^2}\right\}. \tag{3.29}$$

Similarly, from (2.4)

$$P\left\{Y(t)\varepsilon dy(t),\, \theta\varepsilon d\theta \mid I(0)=1,\, I(t)=1,\, Y(0)=y(0)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1(t)} \exp\left\{-\frac{1}{2}\left(y(t) - \theta - \rho_1^t(y(0)-\theta)\right)^2 / \sigma_1(t)^2\right\}$$

$$\times \frac{1}{\sqrt{2\pi}\, v_1(0)} \exp\left\{-\frac{1}{2}\left(\theta - m_1(0)\right)^2 / v_1(0)^2\right\} dy(t)d\theta$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1(t)}\,\frac{1}{\sqrt{2\pi}\, v_1(0)} \exp\left\{-\frac{1}{2}\frac{\left(\theta - \alpha_1(t)\right)^2}{\xi_1(t)^2} - \frac{1}{2}R_1(t)\right\} dy(t)d\theta \tag{3.31}$$

where

$$\xi_1(t)^{-2} = \left[\left[(1-\rho_1^t)^2 / \sigma_1(t)^2\right] + v_1(0)^{-2}\right] \tag{3.32}$$

$$\alpha_1(t) = \xi_1(t)^2 \left[\frac{(1-\rho_1^t)\left[y(t) - \rho^t y(0)\right]}{\sigma_1(t)^2} + \frac{m_1(0)}{v_1(0)^2}\right] \tag{3.33}$$

and

$$R_1(t) = \left[(y(t) - \rho_1^t y(0)) - m_1(0)(1-\rho_1^t)\right]^2 \frac{\xi_1(t)^2}{\sigma_1(t)^2 v_1(0)^2}$$

$$= \left[(y(t) - \rho_1^t y(0)) - m_1(0)(1-\rho_1^t)\right]^2 \Big/ \left[(1-\rho_1^t)^2 v_1(0)^2 + \sigma_1(t)^2\right] \tag{3.34}$$

with

$$\sigma_1(t)^2 = \sigma_1^2 \; (1-\rho_1^{2t})/(1-\rho_1^2).$$

Thus,

$$P\left\{Y(t)\varepsilon dy(t) \mid Y(0)=y(0),\, I(0)=1,\, I(t)=1\right\}$$

$$= \frac{1}{\sqrt{2\pi}\;\tau_1(t)} \exp\left\{-\frac{1}{2}(y(t) - \mu_1(t))^2 / \tau_1(t)^2\right\} \tag{3.35}$$

where

$$\tau_1(t)^2 = \left[(1-\rho_1^t)^2 v_1(0)^2 + \sigma_1(t)^2\right] \tag{3.36}$$

$$= \sigma_1(\infty)^2 \frac{\left[(1-\rho_1^{2t})\sigma_1(\infty)^2 + \tau_1^2\, 2(1-\rho_1^t)\right]}{\sigma_1(\infty)^2 + \tau_1^2}$$

$$= (1-\rho_1^t)\, \sigma_1(\infty)^2 \left[1 + \left(\tau_1^2 + \rho_1^t\, \sigma_1(\infty)^2\right) \Big/ \left(\tau_1^2 + \sigma_1(\infty)^2\right)\right]$$

and

$$\mu_1(t) = \rho_1^t y(0) + m_1(0)\,(1-\rho_1^t) \tag{3.37}$$

$$= \frac{\rho_1^t y(0)\left[\sigma_1(\infty)^2 + \tau_1^2\right] + \left[\tau_1^2 y(0) + \sigma_1(\infty)^2 \mu_1\right](1 - \rho_1^t)}{\sigma_1(\infty)^2 + \tau_1^2}$$

$$= \frac{\left\{\tau_1^2 y(0) + \sigma_1(\infty)^2 \mu_1\right\} + \rho_1^t\left(y(0)\left\{\sigma_1(\infty)^2 + \tau_1^2\right\} - \left[\tau_1^2 y(0) + \sigma_1(\infty)^2 \mu_1\right]\right)}{\sigma_1(\infty)^2 + \tau_1^2}$$

$$= \frac{\tau_1^2 y(0) + \sigma_1(\infty)^2 \mu_1 + \rho_1^t \sigma_1(\infty)^2 \left[y(0) - \mu_1\right]}{\sigma_1(\infty)^2 + \tau_1^2}$$

$$= \frac{\mu_1 \sigma_1(\infty)^2 (1 - \rho_1^t) + y(0)\left[\tau_1^2 + \rho_1^t \sigma_1(\infty)^2\right]}{\sigma_1(\infty)^2 + \tau_1^2} .$$

Similarly,

$$P\left\{Y(t) \varepsilon dy(t) \mid Y(0) = y(0), I(0) = 1, I(t) = 2\right\} = \frac{1}{\sqrt{2\pi}\, \tau_2(t)} \exp\left\{-\frac{1}{2} \frac{\left(y(t) - \mu_2(t)\right)^2}{\tau_2(t)^2}\right\} \qquad (3.38)$$

where

$$\tau_2(t)^2 = \sigma_2(\infty)^2 + \tau_2^2 \qquad (3.39)$$

and

$$\mu_2(t) = \mu_2. \qquad (3.40)$$

Once again we will estimate the identity of the item that was observed at time t to be that item j for which

$$\frac{p_j(t)\, s_j(t)}{\tau_j(t)} \exp\left\{-\frac{1}{2}\left(y(t) - \mu_j(t)\right)^2 / \tau_j(t)^2\right\} \qquad (3.41)$$

is the largest.

Thus,

$$P\left\{\text{choose item 2} \mid Y(0)=y(0)\ ,\ I(0)=1,\ \ I(t)=1\right\} \tag{3.42}$$

$$= P\left\{\frac{p_2(t)s_2(t)}{\tau_2(t)}\exp\left\{-\frac{1}{2}\left(Y(t)-\mu_2(t)\right)^2/\tau_2(t)^2\right\}\right.$$

$$\left. > \frac{p_1(t)s_1(t)}{\tau_1(t)}\exp\left\{-\frac{1}{2}\left(Y(t)-\mu_1(t)\right)^2/\tau_1(t)^2\right\} \,\middle|\, Y(0)=y(0),\ I(0)=1,\ I(t)=1\right\}$$

$$= P\left\{\frac{p_2(t)s_2(t)}{\tau_2(t)}\exp\left\{-\frac{1}{2}\left(c(t)Z+d(t)\right)^2\right\} > \frac{p_1(t)s_1(t)}{\tau_1(t)}\exp\left\{-\frac{1}{2}Z^2\right\}\right\} \tag{3.43}$$

where Z is a standard normal random variable.

$$c(t) = \frac{\tau_1(t)}{\tau_2(t)} \tag{3.44}$$

$$= \frac{\sigma_1(\infty)\sqrt{1-\rho_1^t}}{\sqrt{\sigma_2(\infty)^2+\tau_2^2}}\left[1+\left[\left(\tau_1^2+\rho_1^t\sigma_1(\infty)^2\right)\Big/\left(\tau_1^2+\sigma_1(\infty)^2\right)\right]\right]^{\frac{1}{2}}$$

and

$$d(t) = \frac{\mu_1(t)-\mu_2(t)}{\tau_2(t)} \tag{3.45}$$

$$= \frac{1}{\sqrt{\sigma_2(\infty)^2+\tau_2^2}}\left[\frac{\rho_1^t(y(0)-\mu_1)\,\sigma_1(\infty)^2+(y(0)-\mu_2)\,\tau_1^2+(\mu_1-\mu_2)\,\sigma_1(\infty)^2}{\sigma_1(\infty)^2+\tau_1^2}\right]$$

Hence

$$P\left\{\text{choose item 2} \mid Y(0)=y(0)\ ,\ I(0)=1,\ \ I(t)=1\right\} \tag{3.46}$$

$$= P\left\{-2\ln\left\{\frac{\tau_1(t)\,p_2(t)\,s_2(t)}{\tau_2(t)p_1(t)s_1(t)}\right\} < (1-c(t)^2)\,Z^2-2c(t)d(t)Z-d(t)^2\right\}$$

$$= P\left\{ \frac{d(t)^2}{1-c(t)^2} - 2\ln\left[c(t)\frac{p_2(t)s_2(t)}{p_1(t)s_1(t)}\right] < \left(\sqrt{1-c(t)^2}\,Z - \frac{c(t)d(t)}{\sqrt{1-c(t)^2}}\right)^2 \right\}$$

if $|c(t)| < 1$

If $p_1(t)\, s_1(t) = 0$, then item 2 will always be chosen. Assume $p_1(t)\, s_1(t) > 0$. If $p_2(t)s_2(t)=0$, then clearly item 2 will never be chosen. In what follows assume
$p_1(t)\, s_1(t)$ and $p_2(t)\, s_2(t)$ are both positive. Note that as $t \to 0$, $\tau_1(t) \to 0$, $c(t) \to 0$ and

$$P\left\{\text{choose item 2} \mid Y(0)=y(0)\ ,\ I(0)=1,\ \ I(t)=1\right\} \to 0.$$

Also, as $\tau_2 \to \infty$ and/or $\sigma_2(\infty) \to \infty$, $c(t) \to 0$, $d(t) \to 0$ and the conditional probability of making a wrong decision tends to zero for $t>0$. Note that as $\sigma_1(t) \to \infty$, $c(t) \to \infty$, $d(t) \to \infty$, and the conditional probability of making a wrong decision tends to 1.

The unconditional probability of a wrong classification can be found by taking the expected value of (3.46) with respect to Y(0). It follows from (3.25) that Y(0) has a normal distribution with mean μ_1 and variance $\sigma_1(\infty)^2 + \tau_1^2$. Figures 4 and 5 show the probability of misclassification for the case $p_2(t)\, s_2(t) = p_1(t)\, s_1(t)$, $\tau_1 = \tau_2 = 1$, $\sigma_1 = \sigma_2 = 1$, $\mu_1 = 1$, $\mu_2 = 2$ and $\rho = 0.0, 0.5, 0.8, 0.9, 0.95$ (resp. curves A, B, C, D, E). The integration to evaluate the probability is performed using Gauss-Hermite quadration with 59 points; cf. A. H. Stroud et al. [1066] and Naylor and Smith [1982]. Figure 4 shows the probability of wrong classification. For small times t having a larger ρ results in smaller probabilities of error. The probability of wrong classification is small for small values of t and then increases to a limiting value as $t \to \infty$. The limiting probability of wrong classification increases as ρ increases. Thus, while for small times, having a larger positive ρ decreases the probability of wrong classification, at larger times it increases the probability of wrong classification. In Figure 5, each probability of wrong classification is divided by its asymptotic (as $t \to \infty$) value. For a fixed time t, for a higher ρ, the fraction of the probability of wrong classification is less than or equal to that for a lower ρ. The higher ρ is, the longer it takes for the probability of error to reach its asymptotic value.

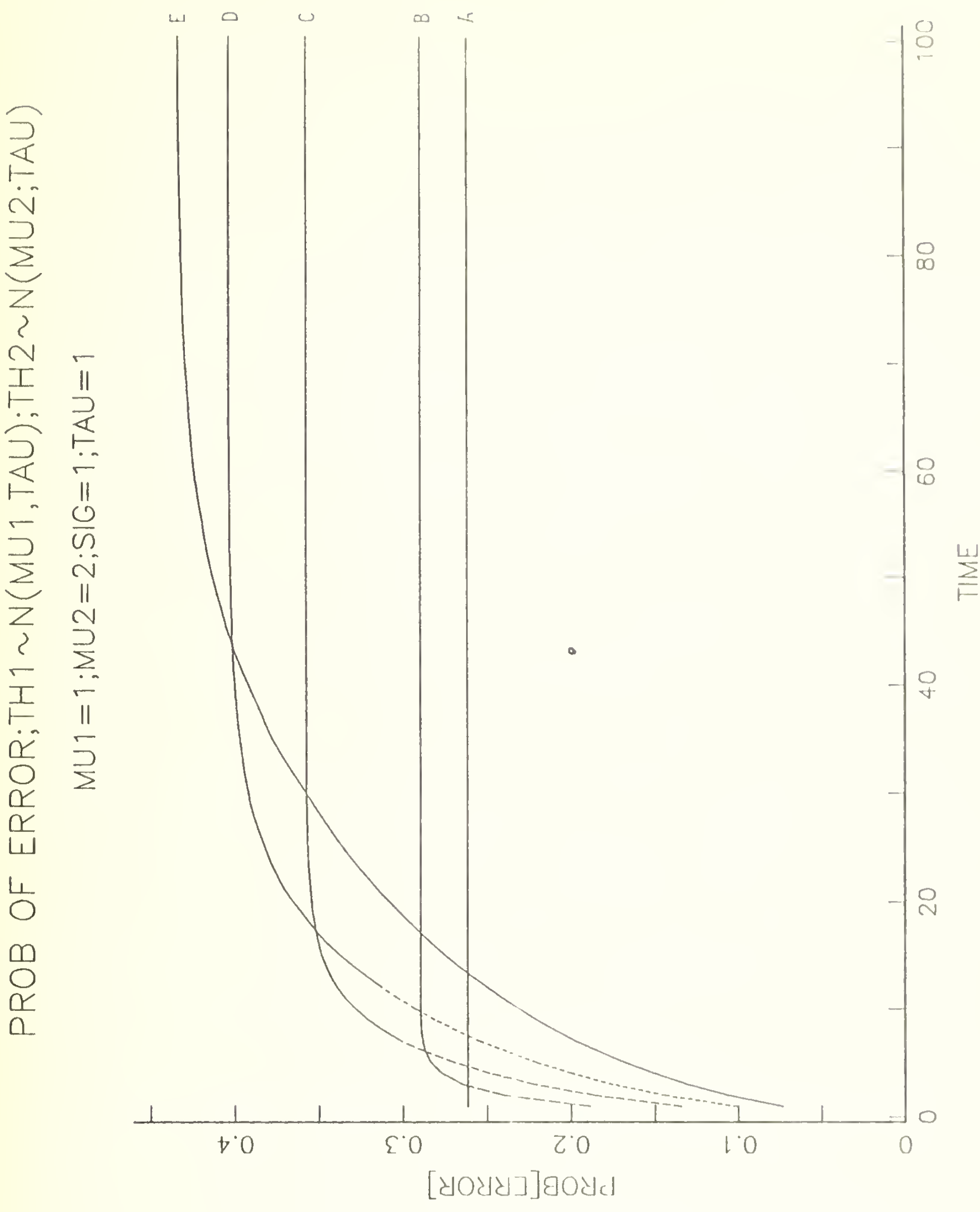
PROB OF ERROR;TH1~N(MU1,TAU);TH2~N(MU2;TAU)
MU1=1;MU2=2;SIG=1;TAU=1
PROB[ERROR]
TIME
0
0.1
0.2
0.3
0.4
0
20
40
60
80
100
A
B
C
D
E

Figure 4

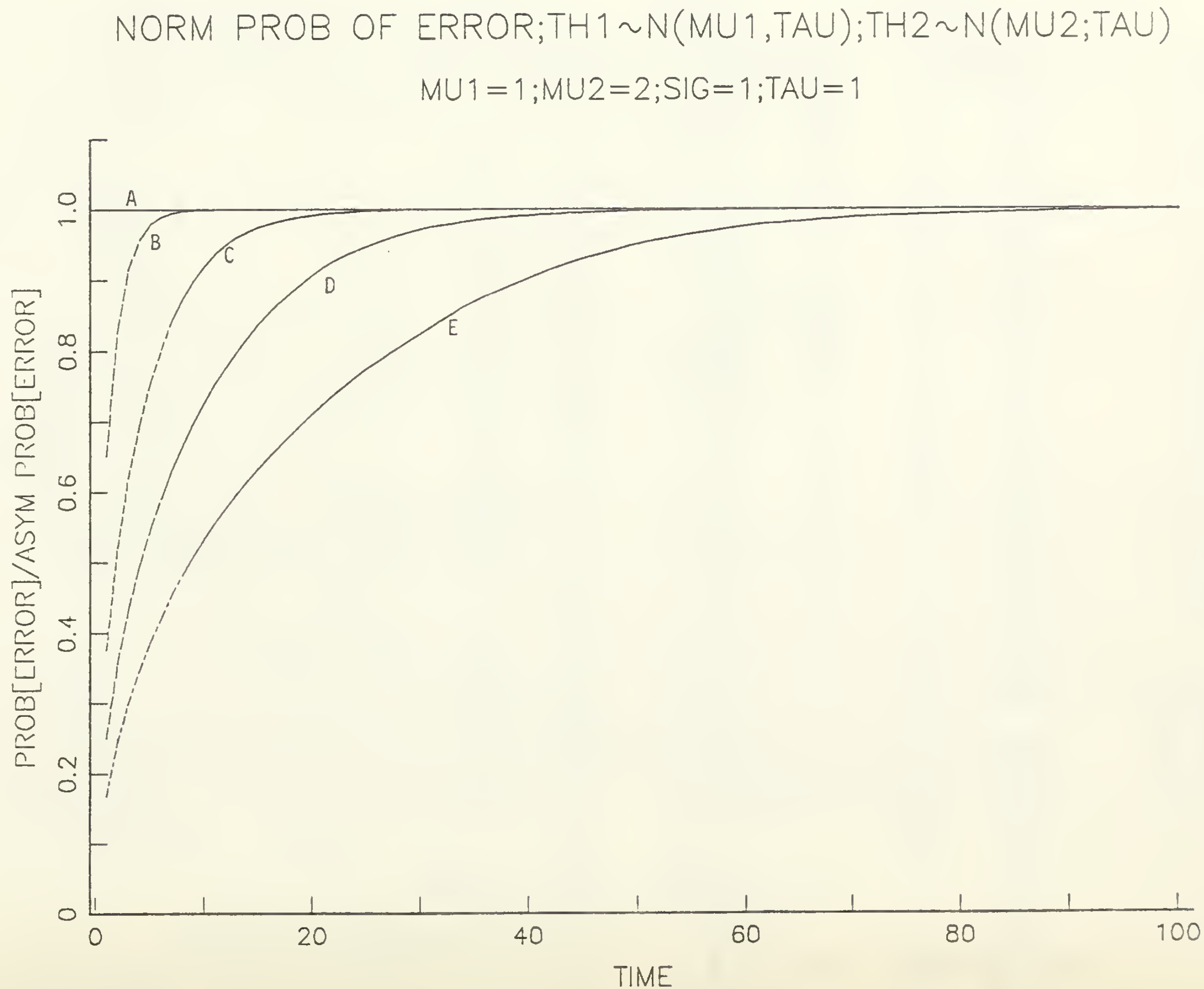
NORM PROB OF ERROR;TH1~N(MU1,TAU);TH2~N(MU2;TAU)
MU1=1;MU2=2;SIG=1;TAU=1
PROB[ERROR]/ASYM PROB[ERROR]
0
0.2
0.4
0.6
0.8
1.0
A
B
C
D
E
0
20
40
60
80
100
TIME

Figure 5

4. Conclusions

This paper considers a model in which there are J items each of which has a characteristic Signature which varies in time. In the examples studied the Signature process of each item is an AR(1)-type process independent of the other items. At time 0 the value of a Signature and the identity of the corresponding item are known. No further values of Signatures are observed until a later time t. At time $t > 0$, a Signature from an unknown item is measured. The problem is to estimate the identity of the item observed at time t. The estimation procedure studied is to estimate the identity of the unknown item to be that one which maximizes the posterior probability of producing the observed signature. The probability of correct identification depends on the parameters of the processes and the magnitude of t. The smaller t is the greater the probability of correct identification of the unknown item. For fixed time t a higher positive correlation results in a lower fraction of the probability of wrong classification with respect to its limiting (as $t \to \infty$) value.

It appears that the problem of this paper resembles the "missing-data problem" of statistics; see Little and Rubin (1987). We plan to investigate the connection more extensively in future work.

5. Acknowldgement

We wish to acknowledge the help of I. O'Muircheartaigh with the Gaussian quadrature procedures used to obtain the figures.

6. References

A. H. Stroud and D. Secrest. *Gaussian Quadrature Formulas.* Prentice-Hall, Inc., Englewood Cliffs, NJ 1966.

J. C. Naylor and A. F. M. Smith, Applications of a method for the efficient computation of posterior distributions. *Appl. Statistics* (1982) **31**, pp. 214-225.

Little, R. J. A., and Rubin, D. B., *Statistical Analysis with Missing Data.* New York: John Wiley and Sons, 1987.

DISTRIBUTION LIST

	NO. OF COPIES
Library (Code 0142) Naval Postgraduate School Monterey, CA 93943-5000	2
Defense Technical Information Center Cameron Station Alexandria, VA 22314	2
Office of Research Administration (Code 012) Naval Postgraduate School Monterey, CA 93943-5000	1
Library (Code 55) Naval Postgraduate School Monterey, CA 93943-5000	1
Operations Research Center, Rm. E40-164 Massachusetts Institute of Technology Attn: R. C. Larson and J. F. Shapiro Cambridge, MA 02139	1
Koh Peng Kong OA Branch, DSO Ministry of Defense Blk 29 Middlesex Road SINGAPORE 1024	1
Arthur P. Hurter, Jr. Professor and Chairman Dept of Industrial Engineering and Management Sciences Northwestern University Evanston, IL 60201-9990	1
Institute for Defense Analysis 1800 North Beauregard Alexandria, VA 22311	1
Professor H. G. Daellenbach Dept of Operations Research University of Canterbury Christchurch, NEW ZEALAND	1
Dept of Operations Research (Code 55) Naval Postgraduate School Monterey, CA 93943-5000	93